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ELASTIC WAVE DAMPING IN THIN-LAYERED SATURATED POROUS MEDIA*

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Monochromatic wave propagation in thin-layered saturated porous media is examined by averaging differential equations with rapidly oscillating coefficients. Particular attention is given to the transformation mechanism for the damping of such waves. Existing results in this area /1/ are extended and refined.

The transformation mechanism /1/ of elastic wave damping in saturated porous media, proposed within the framework of Frenkel'-Biot theory, enabled the anomalous damping at low frequencies to be explained. The basis of the investigations /1/ was the approximate selfconsistent field method. In our opinion, another approach, relying on the well-developed procedure of averaging differential equations with rapidly oscillating coefficients /4, 5/, is possible and preferable. Results obtained using this method have a better foundation from the mathematical point of view and enable the limits of applicability of the formulas in /1/ to be established. The simplest model suitable for a study of the transformation damping mechanics is the model of a layered medium, which is understandably of independent interest.

1. Let the characteristics of a saturated porous medium depend solely on the single coordinate x. We will first study the simplest case when the direction of wave propagation coincides with the x-axis. Independently of the transverse waves, the longitudinal waves are here described, according to /2, 3/, by the following system of equations:

$$P\frac{\partial^2 u}{\partial t^2}+\frac{\mu_0}{k}E_0\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{1}{\beta_0}R\frac{\partial u}{\partial x}\right)=0 \tag{1.1}$$

$$P=\left\|\begin{matrix}p_1 & 1\\ 1 & p_2\end{matrix}\right\|,\quad E_0=\left\|\begin{matrix}1 & 0\\ 0 & 0\end{matrix}\right\|$$

$$R=\frac{\beta_0}{\beta}\left\|\begin{matrix}1 & 1-\varepsilon_0\\ 1-\varepsilon_0 & (1-\varepsilon_0)^2+\beta(\lambda+2\mu)\end{matrix}\right\|=\left\|\begin{matrix}r_{11} & r_{12}\\ r_{12} & r_{22}\end{matrix}\right\|$$

$$\beta=m\beta_1+(1-m-\varepsilon_0)\beta_2,\ \varepsilon_0=\beta_2(\lambda+2\mu/3)$$

$$p_1=(\varkappa m)^{-1},\quad p_2=\rho/\rho_1=(\rho_1 m+\rho_2(1-m))/\rho_1$$

The components u_1, u_2 of the vector function u are here the mean displacment of the liquid phase relative to the solid and the displacement of the solid phase, respectively, ρ is the density, β is the compressibility, m is the porosity, k is the permeability, μ_0 is the liquid phase viscosity, λ and μ are Lamé coefficients for the empty skeleton, and the subscript indicates the phase. The dimensionless parameter $\varkappa\leqslant 1$ is obviously related to the apparent mass coefficients in Biot's theory.

All the characteristics of the medium $p_1, p_2, k, \beta, \varepsilon_0, \lambda, \mu$ are considered to be rapidly oscillating functions of the coordinate x. In other words, thin-layered media are examined. This means that the spatial scale D of the change in medium characteristics is significantly less than the characteristic length $\lambda=f\sqrt{\beta_0\rho_1}$ of a monochromatic wave of the first kind. Here $f=2\pi\omega$ is the frequency, and β_0 is the characteristic value of $\beta(x)$. The specific form of the dependence of the above-mentioned parameters of the medium on x will be refined later. They can be periodic, random functions of x. For the calculations below to be valid it is just essential that these functions "behave identically" in all arbitrarily remote parts of space (/15/, p.378).

The solution of (1.1) corresponding to monochromatic wave propagation is sought in the form

$$u=U(y)\exp[i\omega(t-\sqrt{\beta_0\rho_1}\xi x)] \tag{1.2}$$

where the quantity ω is fixed, $y=x/D$ is the "fast" coordinate, while ξ, U are the desired quantity and the vector-function, respectively. Slower growth at infinity than for a linear function is required of the function U and its derivatives /5/

$$U^{(j)}(y)/|y|\to 0\quad(|y|\to\infty),\ j=0,1,\ldots \tag{1.3}$$

In the case, say, when the dependence of the medium parameters on y is periodic, this

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condition is automatically satisfied after the natural requirement of the periodicity of U.

Taking (1.2) into account, (1.1) reduces to the form

$$(RU')' - iqE_0U - i\xi\varepsilon((RU)' + RU') + \varepsilon^2(PU - \xi^2RU) = 0 \tag{1.4}$$
$$\varepsilon = D\omega\sqrt{\beta_0\rho_1}, \quad q = D^2\omega\beta_0\mu_0/k$$

The physical meaning of the coefficient ε is clarified above, and q is the square of the ratio between the characteristic scale of the medium and the characteristic length of damping of a wave of the second kind.

Using the averaging method /4/, the solution of problem (1.4) is sought as a series in powers of the small parameter ε

$$\xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \ldots \tag{1.5}$$
$$U = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} + \varepsilon\begin{Vmatrix} u_{01} \\ u_{02} \end{Vmatrix} + \varepsilon^2\begin{Vmatrix} u_{11} \\ u_{12} \end{Vmatrix} + \ldots$$

Satisfaction of conditions of the form (1.3) is required of the functions u_{j1} and u_{j2}. Here the $u_{j2}(y)$ are defined to the accuracy of an arbitrary constant that is not essential here. Note that because the principal term of the expansion (1.5) of the relative displacement of the liquid and solid phases equals zero, (1.5) extracts the most intresting case of propagation of waves, on the average, of the first kind for study.

Substitution of (1.5) into the left-hand side of (1.4) with subsequent setting of the coefficients therein to zero for different powers of ε results in a sequence of so-called problems in a cell to determine u_{j1}, u_{j2}, ξ_j:

$$(r_{12}u_{01}' + r_{22}u_{02}')' - i\xi_0r_{22}' = 0 \tag{1.6}$$
$$(r_{11}u_{01}' + r_{12}u_{02}')' - iqu_{01} - i\xi_0r_{12}' = 0$$
$$(r_{12}u_{11}' + r_{22}u_{12}')' - i\xi_1r_{22}' - i\xi_0[(r_{12}u_{01}' + r_{22}u_{02}') +$$
$$(r_{12}u_{01} + r_{22}u_{02})'] - \xi_0^2r_{12} + p_2 = 0, \ldots$$

Only the first three equations of their infinite chain are written down here. They are adequate for finding ξ_0.

In fact, expressing u_{02}' from the first equation in terms of u_{01}'

$$u_{02}' = i\xi_0(1 - Ar_{22}^{-1}) - r_{12}r_{22}^{-1}u_{01}' \tag{1.7}$$

with a still unknown constant A and substituting this expression into the second equation, we find that $u_{01} = -i\xi_0Ag(y)$, where g is the unique solution satisfying conditions of the form (1.3) for problem

$$(\bar{r}_{12} + rg')' - iqg = 0 \quad (\bar{r}_{12} = r_{12}r_{22}^{-1}, r = r_{11} - \bar{r}_{12}) \tag{1.8}$$

Taking the average of both sides of (1.7) and using the equality $\langle u_{02}'\rangle = 0$, that follows from the conditions imposed on u_{02}, we find that $A = \langle r_{22}^{-1} - g'\bar{r}_{12}\rangle$, after which u_{02} is determined from (1.7) by simple integration. The angle brackets here and henceforth denote the mean over the space. Finally, taking the average of both sides of the third equality in (1.6), and using the conditions imposed on u_{11}, u_{12} and the expressions obtained above for u_{01}, u_{02}, we find that

$$\xi_0^2 = \langle p_2A\rangle = \langle p_2\rangle\langle r_{22}^{-1} - g'\bar{r}_{12}\rangle \tag{1.9}$$

Thus, together with u_{01}, u_{02}, the quantity ξ_0 is determined completely by relationships (1.8) and (1.9). Utilization of the last equations of the chain (1.6) enables us in principle, to find ξ_1, u_{11}, u_{12} etc.

However, we will confine ourselves in this paper to studying the principal term ξ_0 of the expansion (1.5) of the wave number ξ. We just note that if the medium is such that the wave numbers of the monochromatic waves propagating in the directions of increasing and decreasing x are equal, then $\xi_1 = 0$ and $\xi = \xi_0 + O(\varepsilon^2)$. The isotropy condition mentioned is perfectly natural and, for instance, is always satisfied for periodic media when the parameters of the medium are even functions of x.

We note that the imaginary part of ξ_0 is generally different from zero. Hence, damping of the wave of the first kind under consideration here on the average already holds in a term of zero-th order of smallness. For a homogeneous medium $\xi_0^2 = p_2r_{22}^{-1}$ is a real number and the situation is quite different. As can be seen /2, 3/, the damping factor α normalized to $\omega\sqrt{\beta_0\rho_1}$ is of the second order of smallness

$$\frac{\alpha}{\omega\sqrt{\beta_0\rho_1}} = \varepsilon^2\frac{1}{2q\sqrt{p_2r_{22}}}(1 - p_2\bar{r}_{12})^2 + O\left(\frac{\varepsilon^4}{q^2}\right)$$

Therefore, the presence of an inhomogeneity results in a significant increase in the damping factor. For a quantitative estimate of this effect it is necessary to solve problem (1.8) in a cell with subsequent calculation of the wave number ξ_0 by means of (1.9).

2. Following Gurevich and Lopatnikov /1/, we first consider the case when the medium parameters differ slightly from their mean values over the space. Here (1.8) is simplified by replacing the functions r and q therein by the spatial means and the problem of finding ξ_0 is solved quite simply for different models of the medium. Indeed assuming that

$$\bar{r}_{12} = \langle \bar{r}_{12} \rangle + \delta G(y) \quad \langle G^2 \rangle = 1$$

and taking into account that the function

$$\exp\left[-(1+i)\nu \mid y - y_0 \mid\right]/[2(1+i)\nu\langle r\rangle]$$

is a fundamental solution of (1.8) for $\nu = (\langle q\rangle/[2\langle r\rangle])^{1/2}$ in which r and q are replaced by their means, we obtain

$$\langle g'\bar{r}_{12}\rangle = \delta^2 \langle r\rangle^{-1} \left\langle 1 - \frac{(1+i)\nu}{2} \int_{-\infty}^{\infty} G(y) G(y_0) \exp - [(1+i)\nu \mid y - y_0 \mid] \, dy_0 \right\rangle \tag{2.1}$$

Let G_I and $-G_R$ denote the imaginary and real parts of the quantity $\langle g'\bar{r}_{12}\rangle$ normalized to $\delta^2 \langle r\rangle^{-1}$. Because of the smallness of the root-mean-square deviation δ these quantities are identical with the damping decrement ϑ and the dispersion of the wave propagation velocity v, apart from a factor which depends only on the mean characteristics of the medium

$$\vartheta = \pi \langle r_{22}^{-1}\rangle^{-1} \langle r\rangle^{-1} \delta^2 G_I \quad (v_\infty - v)/v_\infty = {}^1/_2 \langle r_{22}^{-1}\rangle^{-1} \langle r\rangle^{-1} \delta^2 G_R, \tag{2.2}$$

$$v_\infty = (\langle \rho\rangle^{-1} \langle \lambda + 2\mu + (1 - \varepsilon_0)^2 \beta^{-1}\rangle)^{1/2}$$

Here v_∞ is the velocity of propagation of waves of infinitely high frequency.
Let us make the function G and the values of G_I and G_R obtained for it from (2.1) specific.

Model 1. $G(y) = \sqrt{2}\cos y$. Then

$$G_I = 2\nu^2/(1 + 4\nu^4), \quad G_R = 1/(1 + 4\nu^4) \tag{2.3}$$

Model 2. $G(y)$ is an ergodic stationary random field with correlation function $\langle G(y_0 + y) G(y)\rangle = \exp(-\mid y_0 \mid)$. We have

$$G_I = \nu/(1 + 2\nu + 2\nu^2), \quad G_R = (1 + \nu)/(1 + 2\nu + 2\nu^2) \tag{2.4}$$

Model 3. $G(y) = G_j$ for $j \leqslant y < j + 1$, j is an integer, G_j are independent samples of a random variable with zero expectation and unit variance. In this case

$$G_I = \operatorname{Im}\left[\frac{e^{-(1+i)\nu} - 1}{(1+i)\nu}\right], \quad G_R = \operatorname{Re}\left[\frac{1 - e^{-(1+i)\nu}}{(1+i)\nu}\right] \tag{2.5}$$

The qualitative form of the functions $G_I(\nu)$, $G_R(\nu)$ is identical in all three cases. The function G_R decreases monotonically from unity for $\nu = 0$ to zero for $\nu = \infty$, and G_I vanishes for $\nu = 0, \nu = \infty$. Attention is drawn to the presence of the point $\nu_0 \sim 1$ of the maximum of the function G_I. Consequently, the damping decrement has a definite "selectivity": it is a maximum at the scale inhomogeneities $D_0 = (4\pi \langle r\rangle \langle k\rangle \cdot \nu_0/\beta_0\mu_0 f)^{1/2}$ for waves of frequency f and, conversely, the damping decrement at the scale inhomogeneities D is a maximum for waves of frequency $f_0 = 4\pi \langle r\rangle \langle k\rangle \nu_0/(\beta_0\mu_0 D^2)$.

We note that the model of the medium noted here as Model 2 was taken when computing the damping factors /1/. However, a stronger condition than the condition of smallness of ε was utilized here, namely, in addition to the smallness of ε the smallness of $\nu^2 = \text{const}\, D^2\omega$ was also required in substance. In conformity with this, the formula $G_I = \nu$ /1/ was obtained for Model 2, which is obviously an asymptotic representation of the more general relationship presented above as $\nu \to 0$. In fact, the condition for the smallness of $\nu\ (\nu \ll \nu_0)$ is satisfied only for quite low frequencies. Thus, for values of the parameters taken in /1/ (particularly $D = 0{,}2$ m and $k = 10^{-12}\ \text{m}^2$) the frequency $f_0 = 40$Hz corresponds to the quantity ν_0.

3. The applicability of the results obtained above is limited to the case of media whose spatial dimensions of the inhomogeneities are identical or close to each other. There is obviously a fairly broad spectrum of inhomogeneity scales in real porous media. The difference between the theoretical results (2.3)-(2.5) and known experimental data may be due to this. It is that, as a rule, the experimentally measured damping decrement is constant $(G_I = \text{const})$

over a broad frequency range. Introducing media with a bountiful set of inhomogeneities of different size and equally often encountered, into the consideration enables this to be explained.

Let us formalize the representation of such media in an example of Model 3. We will first generalize it by defining $G(y)$ as follows: $G(y) = G_j$ for $D_j \leqslant y < D_{j+1}$; $D_{j+1} - D_j = d_j$, $D_0 = 0$, $j = \ldots, -1, 0, 1, \ldots$ It was earlier assumed that $d_j = 1$. Here the d_j are considered to be independent samples of the random variable d. In this case, assuming the random variables G_j and d_j to be independent, formulas (2.5) acquire the form (E is the symbol for the expectation value)

$$G_R - iG_I = E(1 - e^{-(1+i)\nu d})/[(1 + i)\nu E d]$$

Let d take the values $\Delta_{-N}, \ldots, \Delta_{-1}, \Delta_0, \Delta_1, \ldots, \Delta_N$. It can be considered that $\Delta_0 = 1$ and Δ_j are enumerated in increasing order. We assume, firstly, that the volume fractions of particles of different dimensions are identical, so that d takes the value Δ_j with probability σ/Δ_j. The quantity σ is determined by the normalization condition. The second assumption is such that all the scales Δ_j are equivalent in the sense that the ratio Δ_{j+k}/Δ_j is independent of the number j. Hence it follows that the Δ_j form a geometric progression $\Delta_j = (1 + \Delta)^j$, $\Delta > 0$, the value of Δ_j is taken with probability $\sigma(1 + \Delta)^{-j}$ and $\sigma = (1 + \Delta)^{-N}\Delta/(1 + \Delta - (1 + \Delta)^{-2N})$.

It is natural to examine the limiting case when $\Delta \to 0$, $N \to \infty$ in such a manner that the length of the greatest piece $(M = (1 + \Delta)^N)$ and least piece (M^{-1}) of inhomogeneity remain fixed. It can be seen that the random variable d here converges weakly in the distribution to the random variable having a probability density function equal to zero outside the segment $[M^{-1}, M]$ and $((M - M^{-1})x^2)^{-1}$ within it. Therefore

$$G_R - iG_I \to \frac{1}{(1+i)(M - M^{-1})\nu} \int\limits_{M^{-1}}^{M} (1 - e^{-(1+i)\nu x}) \frac{dx}{x^2} = \tag{3.1}$$

$$\frac{S((1+i)M^{-1}\nu) - S((1+i)M\nu)}{2\pi \ln M}, \quad S(\nu) = \frac{1 - e^{-\nu}}{\nu} + \int\limits_{\nu}^{\infty} e^{-t}\frac{dt}{t}$$

Henceforth, the previous notation is conserved for the limit values of the functions G_R and G_I.

Graphs of the functions $G_I(\nu)$ are represented in the figure for different M. Such a range of variation of the argument ν is present within which G_I, and of course, the damping decrement also, are practically constant. This interval is fairly broad and includes, say, almost two orders of magnitude in ν for $M = 10$, which is equivalent to four orders in frequency. It broadens in direct proportion to the quantity M as the latter increases, being determined formally by the inequality $M^{-1} \ll \nu \ll M$.

Neglecting terms of order $\nu M^{-1}, \nu^{-1}M^{-1}$ in (3.1) compared with one, we can obtain that within the interval mentioned

$$G_I \approx \pi/(8 \ln M), \quad G_R \approx (^1/_2)(1 - (\ln \nu + c)/\ln M) \tag{3.2}$$
$$c = \gamma + {}^1/_2 \ln 2 - 1 = 0.076$$

It is interesting to note that by virtue of the second of formulas (3.2) the quite slow (logarithmic) growth of the wave propagation frequency occurs in the interval of constancy of G_I as the frequency increases

$$v/v_\infty = \text{const} + \pi^{-2}\vartheta \ln f \tag{3.3}$$

Let us also mention two identities that follow from (3.2)

$$\alpha \partial v/\partial f = \text{const}, \quad \alpha^{-1}\partial v/\partial f = \pi^{-2} v_\infty f^{-2} \tag{3.4}$$

that are understandably satisfied in the extracted range of variation of ν. In the first of them there is the frequency-independent constant of the medium. Let us especially single out the second identity of (3.4), requiring independence from neither the properties of the medium nor from the frequency of the wave being propagated for the complex $f^2 v_\infty^{-1}\alpha^{-1}\partial v/\partial f$.

Analogous results hold even for another model of the medium with a broad set of inhomogeneities of different sizes encountered equally often. It is obtained from Model 1 exactly as the model examined above was obtained from Model 3. Namely, it is assumed that

$$G(y) = \sqrt{\frac{2}{2N+1}} \sum_{n=-N}^{N} \cos\left(\frac{y}{(1+\Delta)^n} + \varphi_n\right)$$

with arbitrary constants φ_n. Values of G_I and G_R corresponding to such a dependence have limits for $\Delta \to 0, N \to \infty$ and fixed $(1+\Delta)^N = M$ as before. It turns out that even this limit satisfies relationships (3.2)-(3.4). It is merely necessary to take $c = \frac{1}{2}\ln 2$ in (3.2).

4.Rejection of the simplifying assumption about the closeness of the parameters of the medium to their mean values over the space results in serious difficulties. And the matter here is not so much of complicating the relations betwen the desired characteristics of the process under consideration ϑ, v and the quantities G_I and G_R or the absence of an analytic solution of problem (1.8) in a cell. It can be confirmed that with the exception of just quite exotic cases (2.2) remain fairly exact. The problem in the cell can be solved numerically. The main difficulty is the lack of information, or of replacing it by equally likely hypotheses about the correlations of the quantities $r, \bar{r}_{12}, q$. A knowledge of these is necessary for making the problem in a cell specific. Without such a specification, we can only hope to obtain the qualitative properties and a priori estimates of the desired quantities.

We will present some such general results concerning the nature of the dependence of G_I, G_R, fG_I on the frequency of the wave being propagated. For real values of the parameters, these quantities determine the damping decrement, the velocity variance, and the damping factor, respectively, as was noted above. The proofs rely on the following representations

$$\delta^2 G_I \langle r\rangle^{-1} = \max_{\psi} \langle -q^{-1}((r\psi')')^2 - q\psi^2 + 2\psi \bar{r}_{12}\rangle \tag{4.1}$$

$$\delta^2 G_R \langle r\rangle^{-1} = \max_{\psi} \langle -r((q^{-1}\psi')')^2 - r^{-1}\psi^2 + 2q^{-1}\bar{r}_{12}'\psi'\rangle$$

The maximum is taken over all functions satisfying conditions of the form (1.3). The representations (4.1) follow directly from (1.8) and (1.9). The extremals in (4.1) are here none other than the imaginary and real part of the solution $g(y)$ of the problem in the cell (1.8).

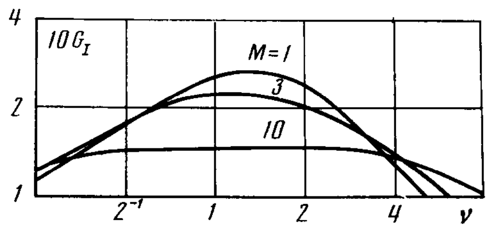

Fig.1

Let the functionals on the right-hand sides of (4.1) be denoted by $J_I(\psi, q)$, $J_R(\psi; q)$.

Property 1.

$$G_I(0) = G_I(\infty) = 0,\ 0 \leqslant G_I \leqslant (1/2)\langle r\rangle/\min(r)$$

We will prove just the upper bound for G_I. The remaining properties follow in an obvious way from (4.1). We rewrite J_I first in a somewhat different form

$$J_1 = \langle -q^{-1}((r\psi')' + q\psi)^2 - 2r(\psi')^2 + 2(\bar{r}_{12} - \bar{r}_{12}\rangle)\psi'\rangle$$

The properties (1.3) of the function ψ were used here for the integration by parts. Discarding the first component on the right-hand side of this equality, we use the quantity

$$\max_{\psi} \langle -2r(\psi')^2 + 2(\bar{r}_{12} - \langle\bar{r}_{12}\rangle)\psi'\rangle$$

as the upper limit of $\max J_I$.

The problem occurring here of seeking the maximum is algebraic in nature. Solving it, we find

$$\max_{\psi} J_I \leqslant 1/2\langle r^{-1}(r_{12} - \langle r_{12}\rangle)^2\rangle$$

from which the estimate mentioned indeed follows.

Property 2. G_R is a positive, monotonically decreasing function of the frequency $G_R(\infty) = 0$ and

$$\langle r\rangle/\max(r) \leqslant G_R(0) \leqslant \langle r\rangle/\min(r)$$

To prove the monotonically of G_2 it is sufficient to establish that $\max J_R(\psi; cq) \leqslant \max J_R(\psi; q)$ for $c \geqslant 1$. The validity of this inequality becomes obvious if it is noted that

$$\max_{\psi} J_R(\psi; cq) = \max_{\psi} J_R(c^{-1}\psi; cq) = \max_{\psi} \langle -r((q^{-1}\psi')')^2 - r^{-1}c^2\psi^2 + 2\psi' q^{-1}(\bar{r}_{12})'\rangle$$

and the last expression is compared with $\max J_R(\psi; q)$.

Property 3. fG_I is a positive monotonically decreasing function of the frequency.

5. We will now turn to the general case of wave propagation in a thin-layered medium when the wave vector makes an angle φ different from zero with the x-axis. Particle displacement occurs in the plane formed by the wave vector and the x-axis. In this case the averaging procedure is completely analogous to that performed earlier. Omitting it, we present just the final results. However, first remaining the case $\varphi = 0$ we mention that

$$\xi^2 = \langle p_2\rangle \langle r_0^{-1}\rangle \; (r_0 = \beta_0\mu)$$

to accuracy $O(\varepsilon^2)$ for transverse waves in the notation taken above so that transverse waves do not damp out with the accuracy mentioned for $\varphi = 0$.

In the general case it is impossible to separate out the longitudinal and transverse waves individually. In their stead there is a pair of waves in each of which the vector of the mean particle displacement makes some angle η with the wave vector that is generally different from zero and $\pi/2$. To accuracy $O(\varepsilon^2)$ this angle and the corresponding wave number are determined from the following system of equations

$$\langle p_2\rangle \xi^{-2}\cos\eta = \cos\eta\,(\langle r_0^{-1}\rangle^{-1} + a_2\sin^2 2\varphi) + \sin\eta\sin 2\varphi\,(a_1 - a_2\cos 2\varphi)$$

$$\langle p_2\rangle \xi^{-2}\sin\eta = \sin\eta\,(A^{-1} + 4a_1\sin^2\varphi - a_2\sin^2 2\varphi) + \cos\eta\sin 2\varphi\cdot(a_1 - a_2\cos 2\varphi)$$

$$a_1 = \langle r_0\rangle - BA^{-1} - (AC - B^2)A^{-1},\quad a_2 = \langle r_0\rangle - \langle r_0^{-1}\rangle^{-1} - (AC - B^2)A^{-1}$$

$$B = \langle r_0 r_{22}^{-1} - \bar{r}_{12} r_0 g'\rangle,\quad C = \langle r_0^2 r_{22}^{-1} - \bar{r}_{12} r_0 h'\rangle$$

Here $h(y)$ is the solution of the equation

$$(\bar{r}_{12} r_0 + rh')' - iqh = 0$$

and the function $g(y)$ and the quantity A have been defined earlier.

It can be shown that this system of equation has two solutions (ξ_p^2, η_p) and (ξ_s^2, η_s) for $0 < \varphi < \pi/2$. The subscripts p and s denote those that describe the longitudinal and transverse waves, respectively, as $\varphi \to 0$. In the general case the imaginary parts of ξ_p and ξ_s do not equal zero. Consequently, unlike the cases $\varphi = 0$, $\varphi = \pi/2$ damping of the zero-th order of smallness that is ensured by the transformation mechanism holds here for both kinds of waves. It can be computed under the additional assumption of a slight difference between the charactersitics of the medium and their mean values over the space. Note that with this assumption the particle displacement in the p-wave occurs principally in the direction of the wave vector and in the s-wave in a perpendicular direction. Consequently, in this case it is natural to call these waves longitudinal and transverse.

Let us define the function $L(y)$ by the relationships

$$r_0(y) = \langle r_0\rangle + \sigma L,\quad \langle L^2\rangle = 1$$

analogous to those utilized above to define G, and let us introduce the quantities L_I, L_R, H_I, H_R in addition to G_I, G_R by the formulas

$$-L_R + iL_I = \left\langle 1 - {}^1\!/_2(1+i)\nu \int L(y)L(y_0)\exp - [(1+i)\nu|y - y_0|]\,dy_0\right\rangle$$

$$-H_R + iH_I = \left\langle G(y)L(y) - ({}^1\!/_2)(1+i)\nu\int G(y)L(y_0)\exp - [(1+i)\nu|y - y_0|]\,dy_0\right\rangle$$

Just as for longitudinal waves at $\varphi = 0$ the damping decrement and the dispersion of the propagation velocity are determined by the functions G_I, G_R, in the case under consideration they are determined by the functions G_I, L_I, H_I and G_R, L_R, H_R, respectively. We present here just the appropriate expressions for the damping decrements

$$\vartheta_s = \pi\langle r_0\rangle\langle r\rangle^{-1}\sigma^2 L_I\sin^2 2\varphi,\quad \vartheta_p = \pi\langle r_{22}^{-1}\rangle\langle r\rangle^{-1}\times[\delta^2 G_I + 4\delta\sigma\langle r_0\rangle\langle r_{22}\rangle^{-1}H_I\sin^2\varphi + 4\sigma^2(\langle r_0\rangle\langle r_{22}\rangle^{-1}\sin^2\varphi)^2 L_I]$$

In particular, they show that the conclusions drawn earlier about the damping decrement of longitudinal waves at $\varphi = 0$ remain valid for both longitudinal and transverse waves in the general case. It is also interesting to note that the maximum of transverse wave damping is reached in the case when the angle between the directions of stratification (the x axis) and wave propagation is 45°.

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DISPERSION OF INTERNAL WAVES BY AN OBSTACLE FLOATING ON THE BOUNDARY SEPARATING TWO LIQUIDS*

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The problem of the scattering of a wave, that propagates along the boundary between two liquids, by a semi-infinite obstacle floating on this boundary is solved in a two-dimensional formulation. The solution is constructed using the Wiener-Hopf method interpreted by Jones in the framework of linear potential theory /1/. The fundamental properties of the processes of scattering and reflection of a wave by the obstacle are stated and an asymptotic analysis of the field in a far zone is presented.

1. We assume tht the half-space $z < 0$ is filled with a homogeneous heavy incompressible liquid of density ρ_1, and the half-space $z > 0$ is filled with a similar liquid of density ρ_2, where $\rho_1 > \rho_2$. Suppose that there are massive particles of some substance floating on the surface $z = 0$ between the liquids, and assume that the particles do not interact with each other as the separating boundary oscillates, or their interactions are negligible. The presence of such particles on the boundary between the liquids enables us to regard the boundary as a massive surface with a surface density of mass distribution $\sigma \geqslant 0$, where σ, being a function of the points of the boundary, may vanish in some of its regions.

We shall confine ourselves to the two-dimensional formulation, and we shall consider the case when the floating substance is contained in the half-plane $\{x > 0,\ z = 0\}$ only, and has constant density σ_0. The half-plane $\{x < 0, z = 0\}$ represents the free separating boundary. We shall ascribe the index 1 to all quantities related to the lower liquid, and the index 2 to those related to the upper liquid.

Suppose that a stationary internal wave of the form

$$u_j^\circ = u_j^\circ(x, z)\exp(-i\omega t) = (-1)^{j+1} A \exp(-a|z| + iax - i\omega t), \quad j = 1, 2$$
$$a = \omega^2(\rho_1 + \rho_2)/[g(\rho_1 - \rho_2)]$$

approaches the massive boundary from infinity along the boundary separating the liquids. Here $u_j^\circ\ (j = 1, 2)$ is the velocity potential and g is acceleration due to gravity.

We shall consider the problem of the diffraction of the internal wave u_j° on the massive part of the boundary. Let us express the amplitude U_j of the velocity potential as the sum

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